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DYNAMIC DEFORMATION OF QUASI-ISOTROPIC COMPOSITE MEDIA

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The problem of a macroscopic description of the dynamics of an elastic composite medium without using assumptions concerning the uniformity of the mean stress-strain states, the magnitudes of the fluctuations and the statistics of the medium parameters [1] is considered. Operator relationships between the mean stress and strain fields allow the application of operator algebra which is well developed in creep theory [2]. It is shown that a quasistatic (matrix) part of the elastic operators, calculated by the method of replacement of variables, yields exact values of the elastic moduli of the composite, where the equations obtained are analogous to the self-consistent field equations [3 — 5]. The interrelation between the mean dimensions of the inhomogeneities and the lengths of the incident and scattered waves is investigated for a specific correlation function.

Analytical computational formulas for the elastic moduli of composite media have been obtained in [3 — 5] on the basis of classical solutions and the self-consistent field method. Exact formulas for the stochastic model have first been obtained in [6] on the basis of a strongly isotropic model, and on the basis of an equivalent singular approximation in [7]. The derivation of exact formulas requires homogeneity of the mean stress-strain states and summation of infinite sequences of the perturbation series (operators, in the general case) in the cases considered.

The method of replacing the field variables by their polarized values turns out to be equivalent to a partial summation of definite kinds of Feynman diagrams. It is established by a direct computation on the basis of the compatibility equations that the macroscopic moduli determined by the equations obtained are exact. A direct analytical comparison with formulas presented in [7, 8] is possible for two-phase composites; numerical computations for certain polycrystals of cubic symmetry yield the same values as the formulas in [6].

Finding dispersion relations for composite media in the general case requires taking account of spatial dispersion, which is equivalent to an exact calculation of the eigenvalues of the polarization and elastic operators. A computation of the single- and two-point approximations by means of the polarization tensor results in summation of sequences of all single- and two-point moments of the elastic parameters. Reduction of the calculation procedure to Weber-Schafheitlin integrals [9] imposes no constraints on the wavelength band for the correlation function selected, which will permit establishment of the discontinuous nature of the dispersion formulas. We obtain the long- and shortwave asymptotics (neglecting spatial dispersion) by using a plane-wave approximation of the Green's tensor and taking account of the first approximation in the parameter $ak \ll 1$ and $ak \gg 1$, respectively.

1. The dynamics of a composite elastic unbounded medium with a harmonic dependence of the field variables on the time is described by the equations

$$Lu = 0, \quad L = \partial\Lambda\partial + \rho_0\omega^2 \tag{1.1}$$

where ω is the frequency, ρ_0 is the density, Λ is the elastic modulus tensor, dependent in a random manner on the space coordinates. The connection between the stress and strain is determined by the relationships

$$\sigma = \Lambda e, \quad e_{i,j} = {}^1/_2\,(u_{i,j} + u_{j,i}) \tag{1.2}$$

In the general case, we write the connection between the mean stress and strain fields in the operator form

$$\langle\sigma\rangle = \langle\Lambda e\rangle = N^*\langle e\rangle = \int\Lambda^*(x - x_1)\langle e(x_1)\rangle\, dx_1 \tag{1.3}$$

Taking account of the relationships (1.1)—(1.3), we obtain that the mean displacement fields satisfy the equations

$$L^*\langle u\rangle = 0, \quad L^* = \partial N^*\partial + \rho_0\omega^2 \tag{1.4}$$

An investigation of the existence conditions for the mean fields as plane waves e^{ikx} (or in their expansions) results in analysis of the relations

$$\| k\,\Pi^*(\omega, k)\,k - \rho_0\omega^2 \| = 0 \tag{1.5}$$

Here Π^* are the eigenvalues of the elastic operator, which when known permits finding the wave damping velocity and coefficients [10], where in the general case spatial dispersion due to inhomogeneity of the medium holds. The procedure to obtain the relationships between the mean stresses and strains usually results in series in powers of the tensor Λ. Consequently, the constraint by the scope of the correlation approximation in Λ generally assumes smallness of the elastic parameter fluctuations.

Let us introduce an equivalent auxiliary homogeneous body with the parameters Λ_0, ρ_0 into the considerations, for which the equations of motion are

$$L_0u_0 = 0, \quad L_0 = \partial\Lambda_0\partial + \rho_0\,\omega^2$$

We determine the elastic moduli Λ_0 of the auxiliary body from the conditions presented below. Subtracting the equations of motion of the homogeneous medium from (1.1), we obtain

$$L_0u' = \partial\Lambda'\partial u, \quad \Lambda' = \Lambda - \Lambda_0, \quad u' = u - u_0$$

Let us pass from the differential to an equivalent integral equation

$$u = u_0 - \int G(x - x_1)\,\partial\Lambda'(x_1)\,\partial u(x_1)\,dx_1$$

Transferring the derivative under the integral sign and differentiating both sides of the relationship, we obtain an integral equation in the strain

$$e = e_0 - \int G,_{xx}(x - x_1)\,\Lambda'(x_1)\,e(x_1)\,dx_1$$

Here $G,_{xx}$ is the second derivative of the dynamical Green's tensor of the auxiliary medium with parameters Λ_0, ρ_0. Representing the derivative of the Green's tensor as the sum of singular and regular parts, we go over to polarized variables [5, 11] by means of the formulas [12]

$$E = Be, \quad \gamma = \Lambda' B^{-1}, \quad B = I + G^{(s)}\Lambda' \tag{1.6}$$

Here Λ_0 is the elastic modulus tensor of the homogeneous auxiliary medium, I is the unit tensor, $G^{(s)}$ is the singular part of the second derivative of the dynamical Green's tensor, and γ is the polarization tensor.

We obtain from (1.1) — (1.6) that the polarized strain field satisfies the integral equation

$$E = e_0 - \int G^{(R)}(x - x_1)\,\gamma(x_1)\,E(x_1)\,dx_1 \tag{1.7}$$

where $G^{(R)}$ is the regular part of the second derivative of the dynamical Green's tensor. Solving (1.7) by successive iterations, we obtain an operator series in powers of the polarization tensor γ. Physically, the assumption of statistical homogeneity and isotropy of the medium is natural

$$\langle \gamma_{ijkl} \rangle = 0 \tag{1.8}$$

Condition (1.8) assures the best convergence of the perturbation series. Taking the average of the series obtained while taking account of (1.8), we verify directly that $\langle E \rangle$ will be a solution of the integral equation

$$\langle E \rangle = e_0 + \iint G^{(R)}(x - x_1)\,Q(x_1 - x_2)\,\langle E(x_2) \rangle\,dx_1\,dx_2 \tag{1.9}$$

Here the operator Q is determined by a series of integral operator convolutions whose kernels are expressed in terms of regular parts of the second derivatives of the Green's tensor and the moments of γ.

On the other hand, taking the average of (1.7), we can write

$$\langle E \rangle = e_0 - \iint G^{(R)}(x - x_1)\,\Gamma^*(x_1 - x_2)\,\langle E(x_2) \rangle\,dx_1\,dx_2 \tag{1.10}$$

Here the kernel Γ^* is introduced by the relationship

$$\langle \gamma E \rangle = P^* \langle E \rangle = \int \Gamma^*(x - x_1)\,\langle E(x_1) \rangle\,dx_1 \tag{1.11}$$

Equating the relationships (1.9) and (1.10), we obtain

$$\Gamma^*(x - x_1) = -Q(x - x_1) \tag{1.12}$$

Taking account of (1.11) there follows from (1.6)

$$\int dx_1 \Gamma^*(x - x_1)\left[I + G^{(s)}\left(\int dx_2 \Lambda^*(x_1 - x_2) - \Lambda_0\right)\right]\langle e(x_1) \rangle = \left[\int dx_1 \Lambda^*(x - x_1) - \Lambda_0\right]\langle e(x_1) \rangle \tag{1.13}$$

The relationships (1.13) permit the eigenvalues of the elastic operator $\Pi^*(\omega, k)$ to be expressed in terms of the eigenvalues D^* (ω, k) of the polarization operator P^*

$$\Pi^* = \Lambda_0 + M^{-1}D^*, \quad M = I - G^{(s)}D^* \tag{1.14}$$

Applying the inverse Fourier transform to (1.14), we obtain an expression for the kernel of the elastic operator Λ^* $(x - x_1)$ in terms of Λ_0 and the kernel Γ^* $(x - x_1)$ of the polarization operator

$$\Lambda^* (\omega, \rho) = \Lambda_0 + (2\pi)^{-3} \int dk\, e^{ik\rho} M^{-1} (\omega, k)\, \Gamma^* (\omega, k) \tag{1.15}$$

Here Λ_0 is the exact value of the static elastic moduli determined from (1.8).

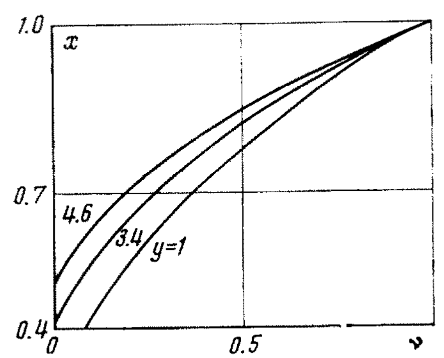

Fig. 1

In fact, by performing analogous calculations on the basis of the dynamical compatibility equations in the stresses, we obtain an equation in Λ_0 from the condition analogous to (1.8) for the stress polarization tensor, which agrees with the equations for Λ_0, obtained on the basis of the equations of motion in displacements. The second term in (1.14) and (1.15) is due, in the general case, to the operator nature of the connection between the mean stress and strain fields in inhomogeneous media.

2. Let us examine the relationships which Λ_0 satisfy for single-phase polycrystals with cubic symmetry. Taking account of (1.6) it follows from (1.8)

$$8G^3{}_0 + G^2{}_0 (9K_0 + 4\mu\nu) - 3G_0\mu (K_0 + 4\mu\nu) - 6K_0\mu^2\nu = 0 \tag{2.1}$$

$$K_0 = \langle K \rangle = \frac{2c_{12} + c_{11}}{3}, \quad \nu = \frac{c_{11} - c_{12}}{2c_{44}}, \quad \mu = c_{44}$$

Here G_0, K_0 are the macroscopic shear and volume elastic moduli, and ν is the anisotropy parameter. Equation (2.1) agrees with the Kröner equation [3, 4] obtained by a consistent method and determines the exact value of G_0 since we obtain the same equation when using the Reuss scheme. A graph of the dependence $x = G_0\mu^{-1}$ on ν is represented in Fig. 1 for different values of $y = K_0\mu^{-1}$.

Let us consider a two-component elastic composite with isotropic components. In this case the relationships (1.8) result in the equations

$$8G^3{}_0 - G_0{}^2 [20 \langle G \rangle - 9K_0 - 12 (G_1 + G_2)] - 3G_0 [5K_0 \langle G \rangle - 2K_0S_g + 4G_1G_2] - 6K_0G_1G_2 = 0, \; (K_0 - \langle K \rangle)(^4/_3G_0 + S_k) + R_k (0) = 0 \tag{2.2}$$

which can also be converted into

$$G_0 = \langle G \rangle - \frac{R_g (0)}{m_0 + S_g}, \quad K_0 = \langle K \rangle - \frac{R_k (0)}{^4/_3 G_0 + S_k} \tag{2.3}$$

$$m_0 = \frac{G_0 (15K_0 + 8G_0)}{6 (K_0 + 2G_0)}, \quad S_f = c_1f_2 + c_2f_1$$

$$c_1 + c_2 = 1, \quad R_f (0) = c_1c_2 (f_1 - f_2)^2$$

Equations in the form of (2.2) have been obtained [5] on the basis of the self-consistent field method, and in the form of (2.3) in [8]. The relationships (2.3) and (2.2) determine the exact values of the elastic moduli, as is verified by a direct computation on the basis of the compatibility equations.

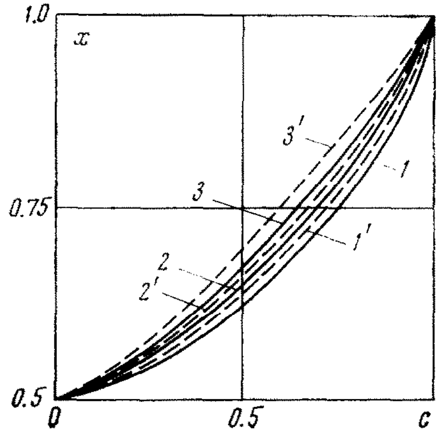

Fig. 2

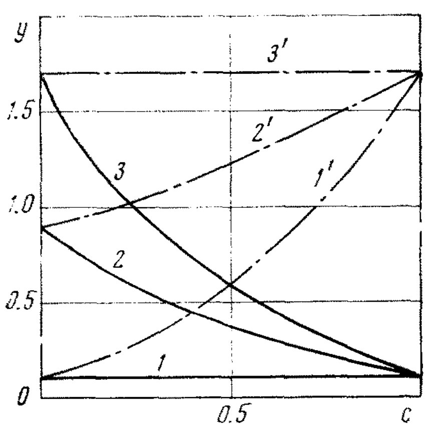

Fig. 3

Shown in Fig. 2 is the dependence of $x = G_0G^{-1}{}_1$ on the concentration $c = c_1$ by solid lines for $\alpha = G_2G^{-1}{}_1 = 0.5$, $\beta = K_1G^{-1}{}_1 = 0.1$ and by dashed lines for $\alpha = 0.5$, $\beta = 1.7$. The dependence of $y = K_0G^{-1}{}_1$ on $c = c_1$ is shown in Fig. 3 for the same values of α, β, δ. Values of the parameter $\delta = K_2G^{-1}{}_1$ equal to 0.1, 0.9, 1.7 correspond to curves 1—3 .

Let us note that the usual assumptions about the homogeneity of the mean state of stress-strain were not used in obtaining (1.8), (2.1), (2.2), (2.3) and no assumption was introduced relative to the statistics of the field of elastic coefficients except the usual hypotheses of statistical homogeneity and isotropy.

3. The discontinuous nature of the dispersion relations for wavelengths on the order of the characteristic scales of the structure in a phenomenological model will hold if the eigenvalues of the elastic operator $\Pi^*(\omega, k)$ in (1.5) depend on k nonuniquely. The exact expression for $D^*(\omega, k)$ in (1.14) is evaluated for the correlation function $\langle\gamma(x_1)\gamma(x_2)\rangle = R(0)\, a\rho^{-1}\sin\rho a^{-1}$ (a is the radius of correlation) by means of the formula

$$D^*(\omega, k) = \int \Gamma^*(\omega, \rho)\, e^{-ik\rho}\, d\rho \qquad (3.1)$$

Going over to a spherical coordinate system in (3.1), we integrate with respect to the angles. We convert the expressions obtained to Weber-Schafheitlin integrals [9], whose values depend on relationships between α and β.

In the case of strong anisotropy of the field Λ for $\overset{*}{\Gamma}_{ijkl}(\omega, \rho)$ by means of (1.12) in the correlation approximation in γ and for $D^*_{ijkl}(\omega, k)$ by means of (3.1), we obtain expressions in the form of isotropic tensors

$$K_{ijkl}(\tau) = K_1(|\tau|)\,\delta_{ij}\delta_{kl} + K_2(|\tau|)\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) +$$
$$K_3(|\tau|)\,(\delta_{nj}e_ke_l + \delta_{kl}e_ne_j) + K_4(|\tau|)\,(\delta_{jl}e_ne_k + \delta_{jk}e_ne_l +$$

$$\delta_{nl}e_je_k + \delta_{nk}e_je_l) + K_5(|\tau|)\,e_ne_je_ke_l, \quad e_i = \tau_i\,|\tau|^{-1}$$

In this case expressions dependent on six functions are obtained for $\Lambda^*(\rho)$ and $N^*(\omega, k)$ by means of (1. 15) and (1. 16). The strong isotropy of $N^*(\omega, k)$ $(N^*_3 = N_4^* = N^*_5 = N_6^* = 0)$ corresponds to the strong isotropy of $D^*(\omega, k)$ $(D^*_3 = D_4^* = D^*_5 = 0)$. In the general case we can write ($J_\gamma(z)$ is the Bessel function)

$$D^*_m(\omega, k) = S_m^{(1)}\left\{\int_0^\infty J_\nu\left[\left(\frac{1}{a} - k_t\right)\rho\right] J_\mu(k\rho)\,\rho^\lambda\,d\rho\right\} +$$

$$S_m^{(2)}\left\{\int_0^\infty J_\nu\left[\left(\frac{1}{a} - k_l\right)\rho\right] J_\mu(k\rho)\,\rho^\lambda\,d\rho\right\} +$$

$$S_m^{(3)}\left\{\int_0^\infty J_\nu\left[\left(\frac{1}{a} + k_l\right)\rho\right] J_\mu(k\rho)\,\rho^\lambda\,d\rho\right\} +$$

$$S_m^{(4)}\left\{\int_0^\infty J_\nu\left[\left(\frac{1}{a} + k_t\right)\rho\right] J_\mu(k\rho)\,\rho^\lambda\,d\rho\right\}, \quad m = 1, ., 5$$

Here $S_m^{(i)}\{\psi_i\}^0$ are sums of integrals of the type ψ_i which depend on the relationship between k and $1/a \pm k_\alpha$ $(\alpha = l,\ t)$. Let us introduce a notation for the integrals: $S_m^{(i)} = S_m^{-(i)}$ for $k < 1/a \pm k_\alpha$; $S_m^{(i)} = S_m^{0(i)}$ for $k = 1/a \pm k_\alpha$; $S_m^{(i)} = S_m^{+(i)}$ for $1/a \pm k_\alpha < k$. We then obtain the following cases for $D^*m(\omega, k)$:

1°. $1/a - k_t > 0$.

1) $k < 1/a - k_t$	$D^*_m = S_m^{-(1)} + S_m^{-(2)} + S_m^{-(3)} + S_m^{-(4)}$
2) $k = 1/a - k_t$	$D^*_m = S_m^{\circ(1)} + S_m^{-(2)} + S_m^{-(3)} + S_m^{-(4)}$
3) $1/a - k_t < k < 1/a - k_l$	$D^*_m = S_m^{+(1)} + S_m^{-(2)} + S_m^{-(3)} + S_m^{-(4)}$
4) $k = 1/a - k_l$	$D^*_m = S_m^{+(1)} + S_m^{\circ(2)} + S_m^{-(3)} + S_m^{-(4)}$
5) $1/a - k_l < k < 1/a + k_l$	$D^*_m = S_m^{+(1)} + S_m^{+(2)} + S_m^{-(3)} + S_m^{-(4)}$
6) $k = 1/a + k_l$	$D^*_m = S_m^{+(1)} + S_m^{+(2)} + S_m^{\circ(3)} + S_m^{-(4)}$
7) $1/a + k_l < k < 1/a + k_t$	$D^*_m = S_m^{+(1)} + S_m^{+(2)} + S_m^{+(3)} + S_m^{-(4)}$
8) $k = 1/a + k_t$	$D^*_m = S_m^{+(1)} + S_m^{+(2)} + S_m^{+(3)} + S_m^{\circ(4)}$
9) $1/a + k_t < k$	$D^*_m = S_m^{+(1)} + S_m^{+(2)} + S_m^{+(3)} + S_m^{+(4)}$

2°. $1/a - k_t < 0$. Then we take the value $z = k_t - 1/a > 0$ as the argument in the expression

$$\sqrt{2z/\pi}\,J_{\pm 1/2}(z)$$

in the formulas for $S_m^{(1)}$, and we take the sign before the $J_{+1/2}(z)$ opposite to the sign in the case 1° $(z = 1/a - k_t > 0)$. Changes in the formulas will occur only for the subcases (1) — (4).

3°. $1/a - k_l < 0$. Analogously to case 2° we change the argument and sign in the expressions for $J_{\pm 1/2}(z)$ and we introduce corresponding changes in formulas (1)—(4).

The remaining expressions for D_m^* are analogous to case 1°.

The values for $S_m^{(i)}$ evaluated by means of the Weber Schafheitlin formulas [9] are expressed by the hypergeometric function $F(\alpha, \beta, \gamma; z)$. The argument z in the expressions for $S_m^{-(i)}$ equals $k(1/a \pm k_\alpha)^{-1}$, and correspondingly $(1/a \pm k_\alpha)k^{-1}$ for $S_m^{+(i)}$, while ambiguity of $D_m^*(\omega, k)$ holds for $z = 1$. Substituting $D_m^*(\omega, k)$ into (1.14) and (1.15), we determine the exact dispersion relations from which the wave velocity and damping can be found [10].

Neglecting spatial dispersion, using the plane-wave approximation of the Green's tensor (the scattered field is studied far from the scattering elements) for long $ak \ll 1$ and short $ak \gg 1$ waves, and assuming the smallness of the fluctuations of the elastic parameters of the medium $(\Lambda_0 = \langle\Lambda\rangle,\ \Pi^* = \Lambda_0 + D^*)$, we obtain expressions analogous to those in [10].

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